On the Craw–Ishii Conjecture

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1 The slogan

For a finite group $G \subset \mathrm{GL}_n(\mathbb{C})$,

$$\begin{array}{ccc} & & \mathbb{C}^n \\ & & \downarrow \\ Y & \longrightarrow & \mathbb{C}^n/G, \end{array}$$

let Y be an *interesting* resolution of singularities of $X := \mathbb{C}^n/G$, eg. minimal resolutions, crepant resolutions.

Our goal is to construct Y as a moduli space of G-equivariant sheaves on $\mathbb{C}^n$.

2 G-constellations

Definition 2.1. For a finite group $G \subset \mathrm{GL}_n(\mathbb{C})$, we say that:

(i) a G-invariant subscheme Z of $\mathbb{C}^n$ is called a G-cluster if $\mathcal{O}_Z \cong \mathbb{C}[G]$, the regular representation of G.

(ii) the G-Hilbert Scheme $G\text{-}\mathrm{Hilb}\,\mathbb{C}^n$ is the fine moduli space of G-clusters.

Craw and Ishii generalised this notion.

Definition 2.2. A G-constellation is a G-equivariant sheaf $\mathcal{F}$ on $\mathbb{C}^n$ with $\mathrm{H}^0(\mathcal{F}) \cong \mathbb{C}[G]$.

The representation ring $R(G)$ of G is $\bigoplus_{\rho \in \mathrm{Irr}\, G} \mathbb{Z}\rho$. Define the GIT stability parameter space

$$\Theta = \left\{ \theta \in \mathrm{Hom}_{\mathbb{Z}}(R(G), \mathbb{Q}) \,\middle|\, \theta(\mathbb{C}[G]) = 0 \right\}.$$

Definition 2.3. For $\theta \in \Theta$, we say that:

(i) a G-constellation $\mathcal{F}$ is *θ-semistable* if $\theta(\mathcal{G}) \geq 0, \forall \mathcal{G} \subset \mathcal{F}$.

(ii) a G-constellation $\mathcal{F}$ is *θ-stable* if $\theta(\mathcal{G}) > 0, \forall \mathcal{G} \subset \mathcal{F}$.

(iii) θ is *generic* if every θ-semistable object is θ-stable.

Theorem 2.4 (King). *The moduli space can be constructed by GIT.*

(i) $\exists$ $\mathcal{M}_\theta$ a quasiprojective scheme which is a coarse moduli space of θ-semistable G-constellations up to S-equivalence.

(ii) For generic θ, the scheme $\mathcal{M}_\theta$ is a fine moduli space of θ-stable G-constellations.

(iii) The scheme $\mathcal{M}_\theta$ is projective over $\mathcal{M}_0$.

Remark 2.5. The scheme $\mathcal{M}_\theta$ may not be irreducible.

Observation 2.6 (Ito and Nakajima). *For $\theta \in \Theta_+$, the G-Hilbert Scheme $G\text{-}\mathrm{Hilb}\,\mathbb{C}^n$ is canonically $\mathcal{M}_\theta$ where*

$$\Theta_+ := \left\{ \theta \in \Theta \,\middle|\, \theta(\rho) > 0 \quad \forall \rho \neq \rho_0 \right\}.$$

Simply the moduli space $\mathcal{M}_\theta$ coincides with $G\text{-}\mathrm{Hilb}\,\mathbb{C}^n$ for a particular choice of GIT parameter θ.

3 Related works

3.1 Surface quotient singularities

For the minimal resolution Y of $\mathbb{C}^2/G$, there are interesting results.

Theorem 3.1 (Ito and Nakamura). *For $G \subset \mathrm{SL}_2(\mathbb{C})$, the minimal resolution of $\mathbb{C}^2/G$ is $G\text{-}\mathrm{Hilb}\,\mathbb{C}^2$.*

Theorem 3.2 (Ishii, Kidoh, …). *For small $G \subset \mathrm{GL}_2(\mathbb{C})$, the minimal resolution of $\mathbb{C}^2/G$ is $G\text{-}\mathrm{Hilb}\,\mathbb{C}^2$.*

3.2 Gorenstein canonical quotient singularities

For $G \subset \mathrm{SL}_3(\mathbb{C})$,

(i) the quotient $X = \mathbb{C}^3/G$ is a canonical Gorenstein singularity.

(ii) X has a crepant resolution (which may not be unique).

Question 3.3. Does a crepant resolution of X have a moduli description?

Theorem 3.4 (Nakamura, Bridgeland, King and Reid). *For $G \subset \mathrm{SL}_3(\mathbb{C})$, the G-Hilbert scheme $G\text{-}\mathrm{Hilb}\,\mathbb{C}^3$ is a crepant resolution of $\mathbb{C}^3/G$. In particular, $G\text{-}\mathrm{Hilb}\,\mathbb{C}^3$ is smooth and irreducible.*

Question 3.5. For the other crepant resolutions?!

Theorem 3.6 (Bridgeland, King and Reid, Craw and Ishii). *Let $G \subset \mathrm{SL}_3(\mathbb{C})$ be a finite group. For generic $\theta \in \Theta$, $\mathcal{M}_\theta$ is a crepant resolution of $\mathbb{C}^3/G$. In particular, $\mathcal{M}_\theta$ is smooth and irreducible.*

Theorem 3.7 (Craw and Ishii). *Let $G \subset \mathrm{SL}_3(\mathbb{C})$ be a finite abelian group. Then any projective crepant resolution of $\mathbb{C}^3/G$ is isomorphic to $\mathcal{M}_\theta$ for some $\theta \in \Theta$, i.e. a moduli space of G-constellations.*

Conjecture 3.8 (Craw and Ishii). *Theorem 3.7 is also true for non-abelian cases.*

3.3 The (Generalised) Craw–Ishii Conjecture

For a finite group G in $\mathrm{GL}_3(\mathbb{C})$, $X = \mathbb{C}^3/G$.
Let $\phi\colon Y \to X$ be a relative minimal model of X, i.e.

(i) K_Y is ϕ-nef.

(ii) Y has only $\mathbb{Q}$-factorial terminal singularities.

(iii) Y is projective over X.

Conjecture 3.9 (The Craw–Ishii Conjecture). *With the notation as above, if Y is smooth, then Y is isomorphic to (an irreducible component of) $\mathcal{M}_\theta$ for some $\theta \in \Theta$.*

Note 3.10. This conjecture is a generalised version of the Conjecture 3.8.

4 Main results

The *quotient of type* $\frac{1}{r}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is the cyclic quotient $\mathbb{C}^n/\mu_r$ where μ_r is the group of rth roots of unity and the action is:

$$\mu_r \ni \epsilon : (x_1, x_2, \ldots, x_n) \mapsto (\epsilon^{\alpha_1} x_1, \epsilon^{\alpha_2} x_2, \ldots, \epsilon^{\alpha_n} x_n).$$

We prove the Craw–Ishii conjecture for some abelian group cases.

Craw–Maclagan–Thomas Theorem

Theorem 4.1 (Craw, Maclagan and Thomas). *Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a finite abelian group and $\theta \in \Theta$ generic. Then $\mathcal{M}_\theta$ has a unique irreducible component Y_θ which contains the torus $T := (\mathbb{C}^\times)^n/G$. Moreover Y_θ satisfies:*

(i) Y_θ is a not-necessarily-normal toric variety which is birational to $\mathbb{C}^n/G$.

(ii) Y_θ is projective over $\mathbb{C}^n/G$, which is given by a variation of GIT.

$$\begin{array}{ccc} Y_\theta & \hookrightarrow & \mathcal{M}_\theta \\ \downarrow & & \downarrow \\ \mathbb{C}^n/G & \hookrightarrow & \mathcal{M}_0 \end{array}$$

Remark 4.2. We call the unique irreducible component Y_θ of $\mathcal{M}_\theta$ the *birational component*. ♦

Main theorem

Theorem 4.3. *For a, b, c with $(a, b) = 1$, let $r = abc + a + b + 1$. Let G be the group of type $\frac{1}{r}(1, a, b)$. Let X denote the quotient $\mathbb{C}^3/G$. Let $\phi\colon Y \to X$ be a relative minimal model of X. Then Y is smooth. Moreover Y is isomorphic to the birational component Y_θ of $\mathcal{M}_\theta$ for some $\theta \in \Theta$.*

Conjecture 4.4. *The moduli space $\mathcal{M}_\theta$ is irreducible.*

The proof uses the following two new notions developed recently:

(i) G-bricks which gives a local description of $\mathcal{M}_\theta$.

(ii) round down functions which is compatible with toric star subdivisions.

5 G-bricks

Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be the finite subgroup of type $\frac{1}{r}(\alpha_1, \alpha_2, \ldots, \alpha_n)$. Nakamura gave a local description of the birational component of $G\text{-}\mathrm{Hilb}\,\mathbb{C}^n$ with introducing a G-graph which is a monomial basis of G-clusters.

(i) A G-brick is a generalised version of the G-graph.

(ii) Simply a G-brick is a Laurent monomial basis of G-constellations on Y_θ.

For each G-brick Γ, we can associate it with:

(i) a torus invariant G-constellation $C(\Gamma)$.

(ii) a semigroup $S(\Gamma)$ in the G-invariant Laurent monomial lattice.

(iii) an affine toric open set $U(\Gamma) = \operatorname{Spec} \mathbb{C}[S(\Gamma)]$.

Proposition 5.1. *Let Γ be a G-brick. Let Y_θ be the birational component in $\mathcal{M}_\theta$. For generic θ, assume that $C(\Gamma)$ is θ-stable. Then there exists an open immersion*

$$U(\Gamma) = \operatorname{Spec} \mathbb{C}[S(\Gamma)] \hookrightarrow Y_\theta \subset \mathcal{M}_\theta.$$

Theorem 5.2. *Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a finite diagonal group and θ a generic GIT parameter for G-constellations. Assume that $\mathfrak{S}$ is the set of all θ-stable G-bricks. The birational component Y_θ of $\mathcal{M}_\theta$ is isomorphic to the not-necessarily-normal toric variety $\bigcup_{\Gamma \in \mathfrak{S}} U(\Gamma)$.*

6 The end

In general, $\mathcal{M}_\theta$ can be very singular, eg. it doesn't need to be normal.